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Optimal Power-Limited Rendezvous with Thrust Saturation

Cyrus J. Pardis*
Worcester Polytechnic Institute, Worcester, Massachusetts 01609
and
Thomas E. Carter†
Eastern Connecticut State University, Willimantic, Connecticut 06226

Very general arguments show that thrust saturation is not helpful in terms of performance in optimal spacecraft trajectories, but these arguments do not provide quantitative information. For optimal power-limited rendezvous problems with thrust saturation and linearized equations of motion, two simple mathematical conditions that are relatively easy to check are presented that may be of value to mission planners. These conditions relate respectively to the problem of reaching a terminal point in a specified time in the presence of thrust saturation and in doing so in a fuel-efficient way. They can be useful in indicating whether or not the flight interval should be lengthened or whether or not additional thrusters should be implemented if possible, and if so how many. For the case of optimal power-limited rendezvous with a satellite in circular orbit, quantitative information is presented for certain boundary conditions. For certain low-thrust saturation levels and boundary conditions, significant improvements in performance and computation can be attained by lengthening the flight time or implementing an additional thruster, if possible. In one example, the use of two thrusters instead of one reduced the mission fuel requirements by nearly 50%.

I. Introduction

LOW-THRUST trajectory and rendezvous missions based on power-limited propulsion systems have been considered for some time.[1–14] Recently, there appears to have been renewed interest in this subject.[15–23]

Power-limited rendezvous with hard constraints on the thrust has recently been studied.[22,23] In fact, Kechichian[23] has considered both upper and lower bounds on the thrust. For linearized problems, e.g.,[8–16,20,22] these hard constraints on the thrust introduce profound changes in the nature of the mathematical optimization problem. Although the optimal linear quadratic problem with hard constraints on the controls, generally called the problem of Letov, has been considered in the control literature,[24–35] the specific form encountered in the fixed-time power-limited rendezvous problems has not been included.

In the present paper we consider the effect of an upper bound on the thrust magnitude in the linearized power-limited rendezvous problem. The existence of a solution to this problem and the fuel consumed in an optimal solution are dependent on the magnitude of this bound. We present a necessary condition for a solution of the problem to exist in terms of the bound on thrust and the boundary conditions. We also present a necessary and sufficient condition for an optimal solution to be fuel efficient in terms of the bound on the thrust and the boundary conditions. These results are called, respectively, the existence condition and the efficiency condition.

These conditions show that existence of a solution if one does not exist, or better fuel economy if a solution does exist, may be attained by raising the bound on the thrust. In fact, the smaller the region of saturation, the better; the very best situation being a totally unsaturated flight. For these reasons preliminary mission requirements should lead to the design of engine characteristics that allow a mission to be performed without thrust saturation.

There is another alternative that can also be considered for long-term cost effectiveness. This is the approach of designing spacecraft and engine thrusters more or less independently of specific missions, but in such a way that multiple thrusters can be fitted on a spacecraft to complete a variety of missions. Preliminary mission requirements would then determine the smallest number of thrusters required to complete a specific mission without saturation of any of the thrusters. This viewpoint will be considered in this paper. If a nominal thruster cannot perform a mission, as indicated by the existence condition, or if the mission can be completed only with saturation of the nominal thruster, one can either increase the flight time or add additional thrusters. Formulas for the minimum number of power-limited thrusters necessary to reach the boundary conditions, and to reach them with the least fuel expenditure, for a given flight time are presented based on the two conditions mentioned.

For rendezvous near circular orbit using the Clohessy–Wiltshire equations we display flight paths for various bounds on the thrust magnitude and identical boundary conditions for each simulation. The dependence of the fuel consumption on the thrust bound and on the flight interval is presented graphically. For a specified low-thrust bound, the efficiency condition demonstrates that an additional thruster is required to perform the rendezvous mission efficiently. Fuel consumption is shown to be reduced by nearly 50% through the use of this additional thruster.

II. Mathematical Analysis

We begin with consideration of a very general problem.

A. General Problem

We let θ_0 and θ_f be real numbers where $\theta_0 < \theta_f$ and m is a positive integer. A specified set of functions mapping the bounded interval $\theta_0 \leq \theta \leq \theta_f$ into the Euclidean m-space will be denoted $\mathcal{U}(\theta_0, \theta_f)$ and called the set of admissible controls. An arbitrary element u of this set is called an admissible control.

The function J mapping $\mathcal{U}(\theta_0, \theta_f)$ into the set of nonnegative real numbers is called the cost function or performance index. The general problem is to minimize $J(u)$ over $\mathcal{U}(\theta_0, \theta_f)$ subject to the additional condition that the elements of $\mathcal{U}(\theta_0, \theta_f)$ satisfy a specified dynamic constraint, which in this case is the differential equations that follow from the equations of motion of a spacecraft and the appropriate boundary conditions.

In all that follows we denote the Euclidean norm of a vector (e.g., the square root of the sum of the squares) by the symbol $|\cdot|$.

In much of the mathematical literature it is conventional to assume that the elements of the set of admissible controls are Lebesgue

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*Assistant Professor, Department of Mathematical Sciences.
†Professor, Department of Mathematics and Computer Science. Member AIAA.

measurable. We will settle for somewhat less generality in this paper and require that all admissible controls be piecewise continuous. This avoids the need for the Lebesgue integral and increases the readability of the paper.

We consider the following types of sets of admissible controls. The set of all piecewise continuous functions mapping the closed interval $\theta_0 \le \theta \le \theta_f$ into Euclidean m-space is called $\mathcal{U}_\infty(\theta_0, \theta_f)$. A problem with this set of admissible controls is called a problem with unbounded controls. If we let b denote a positive real number, then

$$\mathcal{U}_b(\theta_0, \theta_f) = \{u \in \mathcal{U}_\infty(\theta_0, \theta_f) \mid |u(\theta)| \le b\}$$

A problem with $\mathcal{U}_b(\theta_0, \theta_f)$ as its set of admissible controls is called a problem with bounded controls.

The following facts are immediate: If solutions exist to the optimization problems over each of the following sets of admissible controls, then since $\mathcal{U}_b(\theta_0, \theta_f) \subset \mathcal{U}_\infty(\theta_0, \theta_f)$, we have

$$\min_{u \in \mathcal{U}_\infty(\theta_0,\theta_f)} J(u) \le \min_{u \in \mathcal{U}_b(\theta_0,\theta_f)} J(u)$$

Moreover, if $b < c$, then

$$\min_{u \in \mathcal{U}_c(\theta_0,\theta_f)} J(u) \le \min_{u \in \mathcal{U}_b(\theta_0,\theta_f)} J(u)$$

Similarly, if $\theta_{f_1} < \theta_{f_2}$, then

$$\min_{u \in \mathcal{U}(\theta_0,\theta_{f_2})} J(u) \le \min_{u \in \mathcal{U}(\theta_0,\theta_{f_1})} J(u)$$

These facts indicate that it is advantageous to raise the bound on the control or increase the duration of the flight, if permissible. How advantageous depends on the particular dynamic system and performance index under consideration. In the following work, we consider a linear quadratic problem representing power-limited rendezvous of a spacecraft subject to linear differential equations, bounded thrust, and fixed flight time. Here the thrust is used as the control function. This work is then applied to the specific rendezvous of a spacecraft near a circular orbit. For specified initial and terminal conditions, the variation of the minimum cost with the bound on thrust magnitude and on the length of the flight interval is presented graphically. From this type of information we can determine the improvement in cost to be gained by implementing an additional thruster or lengthening the flight interval. We emphasize that although the study presented is very specific, the underlying ideas presented above are very general, and a similar analysis should be applicable to problems in which the equations of motion are nonlinear.

B. Linear Problem

Assuming that the equations of motion of a spacecraft can be represented by linear differential equations, we have

$$y'(\theta) = A(\theta)y(\theta) + B(\theta)u(\theta) \tag{1}$$

where u is an admissible control, B is an $n \times m$ matrix-valued function, and A is an $n \times n$ matrix-valued function, both continuous in θ. The prime denotes differentiation with respect to the real variable θ. The n-dimensional vector $y(\theta)$ represents the state of the spacecraft. We shall assume that the system (1) is completely controllable. If the problem is a rendezvous problem, we have the fixed end conditions

$$y(\theta_0) = y_0, \qquad y(\theta_f) = y_f \tag{2}$$

where y_0 and y_f are specified points in n-space. For the types of applications considered, the change in mass due to fuel consumption over the flight interval is assumed to be small compared with the mass of the spacecraft. For this reason the spacecraft mass is considered constant, and the applied acceleration therefore varies directly with the thrust.

Associated with the linear system (1) is its adjoint system

$$\lambda'(\theta) = -A(\theta)^T\lambda(\theta) \tag{3}$$

We let $\Psi(\theta)$ denote any fundamental matrix solution associated with Eq. (3). It is known from the theory of linear differential equations and utilized in previous work[22,36] that any solution of Eq. (1) satisfying the initial condition in Eq. (2) can be written as

$$y(\theta) = \Psi(\theta)^{-T}\left[\Psi(\theta_0)y_0 + \int_{\theta_0}^{\theta} \Psi(\tau)^T B(\tau)u(\tau)\,\mathrm{d}\tau\right] \tag{4}$$

where the superscript T denotes the transpose of a matrix and $-T$ denotes the transpose of its inverse.

To shorten the notation, we let

$$R(\theta) = \Psi(\theta)^T B(\theta) \tag{5}$$

and, as in previous work,[36] introduce the pseudostate vector

$$z(\theta) = \Psi(\theta)^T y(\theta) - \Psi(\theta_0)^T y_0 \tag{6}$$

Letting

$$z_f = \Psi(\theta_f)^T y_f - \Psi(\theta_0)^T y_0 \tag{7}$$

the state vector (4) and its end conditions (2) can be replaced, respectively, by

$$z(\theta) = \int_{\theta_0}^{\theta} R(\tau)u(\tau)\,\mathrm{d}\tau \tag{8}$$

$$z(\theta_f) = z_f \tag{9}$$

It has been shown repeatedly[1,2,13–15,18,19,23] that if $u(\theta)$ represents the thrust or applied acceleration of the spacecraft, then an appropriate index of performance for power-limited propulsion systems is

$$J(u) = \int_{\theta_0}^{\theta_f} u(\theta)^T u(\theta)\,\mathrm{d}\theta \tag{10}$$

Combining Eqs. (8) and (9), we state the optimal bounded-thrust linear power-limited rendezvous problem as follows: Determine $u \in \mathcal{U}_b(\theta_0, \theta_f)$ to minimize the cost function (10) subject to the constraint

$$\int_{\theta_0}^{\theta_f} R(\theta)u(\theta)\,\mathrm{d}\theta = z_f \tag{11}$$

If u is a solution of this problem, then there exist multipliers $l_0 \ge 0$ and $l \in \mathbb{R}^n$, not both zero, such that

$$H(\theta, u(\theta)) = \min_{|v| \le b} H(\theta, v), \qquad \theta_0 \le \theta \le \theta_f$$

where

$$H(\theta, v) = l_0 v^T v + l^T R(\theta)v \tag{12}$$

Appearing in this expression is the well-known primer vector

$$p(\theta) = R(\theta)^T l \tag{13}$$

originally named by Lawden[38] and taking the present form whenever the equations of motion are linear and the cost function is given by expression (10).[22]

If $l_0 = 0$, the problem is called abnormal. If the problem is abnormal, then the assumption of complete controllability implies that the primer vector is not identically zero.[22] If $A(\theta)$ and $B(\theta)$ are analytic, then $p(\theta)$ has at most finitely many zeros in this case, so

$$u(\theta) = -\frac{p(\theta)}{|p(\theta)|}b \tag{14}$$

except at possibly finitely many points.

If $l_0 \ne 0$, the problem is called normal, and l_0 can take any positive value. For convenience, we set $l_0 = 1/2$, and Eq. (12) is minimized by

$$u(\theta) = \begin{cases} -p(\theta) & |p(\theta)| \le b \\ \dfrac{-p(\theta)b}{|p(\theta)|} & |p(\theta)| > b \end{cases} \tag{15}$$

Since the problem is linear and the integrand in the cost is independent of the state vector, it is known that solutions of this form satisfying Eq. (11) are optimal.[37]

A normal solution is called saturated on an interval if $|p(\theta)| > b$ on that interval. It is called unsaturated on an interval if $|p(\theta)| \le b$ on that interval. It is called fully saturated or fully unsaturated respectively if the interval is $\theta_0 \le \theta \le \theta_f$.

1. Boundary Value Problem

We emphasize that an optimal solution u depends on l. Using Eq. (13), we rewrite Eq. (15) as

$$u(\theta) = -h(l,\theta)R(\theta)^T l \tag{16}$$

where

$$h(l,\theta) = \begin{cases} 1 & |R(\theta)^T l| \le b \\ \dfrac{b}{|R(\theta)^T l|} & |R(\theta)^T l| > b \end{cases} \tag{17}$$

Using Eq. (16), the boundary condition (11) becomes

$$N(l,\theta_0,\theta_f)l = -z_f \tag{18}$$

where

$$N(l,\theta_0,\theta_f) = \int_{\theta_0}^{\theta_f} h(l,\theta)R(\theta)R(\theta)^T\,\mathrm{d}\theta \tag{19}$$

The two-point boundary value problem becomes the problem of solving the nonlinear equation (18) for l. Certain solutions of this equation have been computed numerically for the case of rendezvous near circular orbit, and the resulting flight paths are presented graphically in the next section.

For those boundary conditions for which the solution is fully unsaturated, a tremendous simplification occurs and the problem can be solved in closed form. In this case $h(l,\theta) = 1$, $\theta_0 \le \theta \le \theta_f$, and Eqs. (18) and (19) become, respectively,

$$M(\theta_0,\theta_f)l = -z_f \tag{20}$$

where

$$M(\theta_0,\theta_f) = \int_{\theta_0}^{\theta_f} R(\theta)R(\theta)^T\,\mathrm{d}\theta \tag{21}$$

It is known that the matrix $M(\theta_0,\theta_f)$ is invertible if and only if the system (1) is completely controllable. Assuming controllability of this system, we obtain the solution of the boundary value problem

$$l = -M(\theta_0,\theta_f)^{-1}z_f \tag{22}$$

and the optimal solution (16) becomes

$$u(\theta) = R(\theta)^T M(\theta_0,\theta_f)^{-1}z_f \tag{23}$$

2. Existence Condition

If there exists a solution to the optimization problem, that is, if there exists an optimal admissible control u such that the boundary point z_f can be reached, then the minimum cost (10) is between the cost of a fully unsaturated solution and the cost of a fully saturated solution. The cost of a fully unsaturated solution is found by substituting Eq. (23) into Eq. (10) and utilizing Eq. (21). The cost of a fully saturated solution comes from setting $|u(\theta)| = b$, $\theta_0 \le \theta \le \theta_f$, in Eq. (10). The result is that, for any solution, we have the inequality

$$z_f^T M(\theta_0,\theta_f)^{-1}z_f \le J(u) \le b^2(\theta_f - \theta_0) \tag{24}$$

The left side of this expression shows that for given θ_0 and θ_f there exists a positive number c such that

$$J(u) \ge c|z_f|^2 \tag{25}$$

This shows that missions become very expensive in terms of fuel expenditure as $|z_f|$ becomes large. From inequality (24) we also obtain the following existence condition:

$$\frac{z_f^T M(\theta_0,\theta_f)^{-1}z_f}{\theta_f - \theta_0} \le b^2 \tag{26}$$

This condition is necessary for a solution to the problem to exist.

The condition (26) shows that given θ_0 and θ_f, the boundary point z_f cannot be reached if the bound b is too low. In fact, we have the following corollary: If

$$b < \left(\frac{z_f^T M(\theta_0,\theta_f)^{-1}z_f}{\theta_f - \theta_0}\right)^{\frac{1}{2}}$$

then a solution to the problem cannot exist.

3. Efficiency Condition

As previously indicated, the value of the cost function without bounds on the controls is always less than or equal to the value with the inclusion of the bounds for the same boundary conditions and flight interval. In fact, computations presented in the following section indicate that the cost increases sharply with decreasing bounds on the thrust magnitude. Considerations of cost efficiency therefore indicate that thrust magnitude bounds should be as high as possible, resulting in as little thrust saturation as possible. The ideal is a totally unsaturated solution. A necessary and sufficient condition for an optimal solution to be fully unsaturated is that its maximum primer vector magnitude be less than or equal to the bound b. If follows from Eqs. (13) and (22) that a necessary and sufficient condition for a solution to be fully unsaturated is

$$\max_{\theta_0 \le \theta \le \theta_f} \left|R(\theta)^T M(\theta_0,\theta_f)^{-1}z_f\right| \le b \tag{27}$$

This is called the efficiency condition.

The maximum improvement in the cost can be expected if the bound can be raised enough to satisfy Eq. (27).

If we divide inequality (24) by $\theta_f - \theta_0$, then use Eqs. (10) and (23) in the evaluation of $J(u)$, and estimate this integral, we establish

$$\frac{z_F^T M(\theta_0,\theta_f)^{-1}z_f}{\theta_f - \theta_0} \le \max_{\theta_0 \le \theta \le \theta_f} \left|R(\theta)^T M(\theta_0,\theta_f)^{-1}z_f\right|^2 \le b^2 \tag{28}$$

This condition is necessary for the existence of an unsaturated solution. If the right side of Eq. (28) is violated, saturation occurs and fuel efficiency may be lost. If b^2 is further lowered until it is less than the left side of Eq. (28), then the boundary point z_f cannot be attained on the interval $\theta_0 \le \theta \le \theta_f$ and a solution does not exist.

4. Multiple Thrusters

Given a boundary value z_f, if either of the conditions (26) or (27) is violated, there are two approaches to restoring the inequality. The first is to increase the flight duration by raising the value of θ_f. The reader should note that this may change the value of z_f according to Eq. (7). The second is to raise the bound on the thrust magnitude b.

Even though the bound on a thruster is fixed, one can effectively raise the total thrust magnitude bound by consideration of multiple thrusters.

Let ν be a positive integer, and suppose that the effect of mounting ν identical thrusters on a spacecraft can be represented by the replacement of $u(\theta)$ by $\nu u(\theta)$ in Eq. (1). This leads to the eventual replacement of $u(\theta)$ by $\nu u(\theta)$ in Eq. (11). There is no need to replace Eq. (10) by $\nu^2 \int_{\theta_0}^{\theta_f} u(\theta)^T u(\theta)\,\mathrm{d}\theta$ because this expression is minimized if and only if expression (10) is minimized.

The ν optimization problem may be stated as follows: Find $u \in \mathcal{U}_b(\theta_0,\theta_f)$ to minimize expression (10) subject to the constraint

$$\nu \int_{\theta_0}^{\theta_f} R(\theta)u(\theta)\,\mathrm{d}\theta = z_f \tag{29}$$

Clearly, all the preceding results hold upon replacing z_f by z_f/ν. This replacement is performed in Eqs. (18), (20), (22), and (23). The existence condition (26) becomes

$$\frac{z_f^T M(\theta_0, \theta_f)^{-1} z_f}{\theta_f - \theta_0} \leq (\nu b)^2 \tag{30}$$

and the efficiency condition (27) becomes

$$\max_{\theta_0 \leq \theta \leq \theta_f} \left| R(\theta)^T M(\theta_0, \theta_f)^{-1} z_f \right| \leq \nu b \tag{31}$$

Based on these considerations, we see that the smallest number of thrusters necessary to reach the boundary point z_f on the interval $\theta_0 \leq \theta \leq \theta_f$ is the smallest integer ν such that

$$\nu^2 \geq \frac{z_f^T M(\theta_0, \theta_f)^{-1} z_f}{b^2(\theta_f - \theta_0)} \tag{32}$$

Similarly, the smallest number of thrusters for minimum fuel consumption is the smallest integer ν such that

$$\nu \geq \max_{\theta_0 \leq \theta \leq \theta_f} \left| R(\theta)^T M(\theta_0, \theta_f)^{-1} z_f / b \right| \tag{33}$$

If ν is picked to satisfy inequality (33), then the ν optimization problem is solved in closed form. The optimal solution for each thruster follows by adjusting Eq. (23) to

$$u(\theta) = R(\theta)^T M(\theta_0, \theta_f)^{-1} z_f / \nu \tag{34}$$

With some modification, this solution can be made into a feedback law.[20]

III. Power-Limited Rendezvous Near Circular Orbit

We apply the preceding work to the problem of rendezvous of a spacecraft with a satellite in circular orbit.

A. Specific Equations

We consider a satellite in circular orbit about a planet. Let R_c denote the radius of the circle, and the angle of revolution of the satellite at an arbitrary time is θ. The initial and terminal values are θ_0 and θ_f, respectively. The applied thrust and the position of a spacecraft of mass m_0 relative to the satellite are denoted respectively by the vectors $u(\theta)$ and $x(\theta)$ in Euclidean 3-space. Their components in a rotating coordinate frame fixed in the satellite are indicated by the subscripts 1, 2, and 3, referring respectively to outward radial, reverse tangential, and the completion of a right-handed system. The equations of motion of the spacecraft relative to the satellite are described by the well-known Hill–Clohessy–Wiltshire equations:

$$\begin{aligned} x_1''(\theta) &= 2x_2'(\theta) + ku_1(\theta) \\ x_2''(\theta) &= -2x_1'(\theta) + 3x_2(\theta) + ku_2(\theta) \\ x_3''(\theta) &= -x_3(\theta) + ku_3(\theta) \end{aligned} \tag{35}$$

where $k = R_c^3/(\mu m_0)$ and μ is the universal gravitational constant times the mass of the planet.

We now normalize Eqs. (35) by dividing by k, defining the new position variable to be $x(\theta)/k$. This is equivalent to setting $k = 1$ in Eqs. (35), and the variables are considered nondimensional. Any desired units for the results are obtained by multiplication by k, where k is specified in terms of the desired units.

Putting these normalized equations in state vector form, we obtain Eq. (1), where $n = 6$, $m = 3$, and $B(\theta)$ is a constant 6×3 matrix whose top three rows consist only of zeros and whose bottom three rows comprise the 3×3 identity matrix. The 6×6 matrix $A(\theta)$ is constant, given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \tag{36}$$

Using this matrix, we can solve the adjoint system (3) and find a fundamental matrix solution associated with it:

$$\Psi(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -6\theta & -2 & 3\sin\theta & -3\cos\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin\theta & \cos\theta \\ 3\theta & 1 & -2\sin\theta & 2\cos\theta & 0 & 0 \\ 2 & 0 & -\cos\theta & -\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta & -\sin\theta \end{bmatrix} \tag{37}$$

From this and the specified end conditions (2), we can compute the pseudostate boundary point z_f from Eq. (7). Transposing Eq. (5), we get $R(\theta)^T = B(\theta)^T \Psi(\theta)$, from which we see that the 3×6 matrix $R(\theta)^T$ is simply the bottom three rows in Eq. (37). This enables us to determine the matrix $M(\theta_0, \theta_f)$ from Eq. (21) in closed form. To save space we do not write this matrix here because it and even its inverse have been presented in previous work.[20]

We now have enough information to check the existence condition (26) and the efficiency condition (27). If either is violated, we increase the flight terminal angle θ_f if possible or add another thruster if possible.

If the efficiency condition (27) is satisfied, then the optimal solution is completely given in closed form through Eq. (23). If the efficiency condition is not satisfied and we can add new thrusters, the minimum number of thrusters needed for fuel efficiency is the smallest integer ν satisfying Eq. (33). The optimal sclution for each thruster is then determined in closed form through Eq. (34). If we do not add thrusters, the solution will have intervals of saturation and in general will be less fuel efficient. Additionally, the solution of the problem is no longer available in closed form. The matrix $R(\theta)^T$ is taken from the bottom three rows of Eq. (37) and, through Eqs. (17) and (19), defines $N(l, \theta_0, \theta_f)$. The nonlinear equation (18) must then be solved iteratively for l. The optimal solution is then given by Eqs. (16) and (17). For a variety of flight intervals and boundary values, this computation has been performed, and some of the results are presented graphically in the figures that follow.

B. Computer Simulation

Figure 1 demonstrates the nature of an optimal solution having large intervals of saturation. The problem depicted is that of a spacecraft initially 0.2 normalized units above a satellite in circular orbit having the same initial velocity as the satellite, attempting to rendezvous with it in exactly one revolution ($\theta_0 = 0, \theta_f = 2\pi$). The spacecraft has a severe thrust magnitude bound of $b = 0.235$ normalized units. The objective of the rendezvous is for the spacecraft to match the position and velocity of the satellite at the end of one revolution.

For these boundary conditions the numerical solution of Eq. (18) defines the primer vector (13), whose locus is presented in Fig. 1a. The circle of radius $b = 0.235$ results from the bound on the magnitude of the thrust. For a point on the primer vector locus inside this circle, the thrust vector is equal in magnitude and located directly opposite of the point. For a point on the primer vector locus outside the circle, the thrust vector is in the opposite direction of the point but constrained to lie on the circle. This situation is well illustrated in Fig. 1b, which shows the locus of the thrust vector. The curve begins on the extreme right at $\theta = \theta_0$ and moves clockwise, constrained on the circle of radius b, until it branches upward on the left side of the figure through an unsaturated interval inside the circle, moving clockwise until it again reaches the circle boundary at the right, again following the same circular arc but in this case not branching upward, and finally reaching the extreme left side at $\theta = \theta_f$. The magnitude of the thrust vector is presented in Fig. 1c. From this figure the approximate locations of the saturation intervals can be found. The approximate cost of this highly saturated rendezvous is $J = 0.325$. Removing the bound on b, we found that a fully unsaturated solution of the identical problem yielded an approximate cost of $J = 0.166$, somewhat more than half as much.

For this rendezvous problem, we examined the effects of raising the bound b on the flight path and the cost. Figure 2 shows

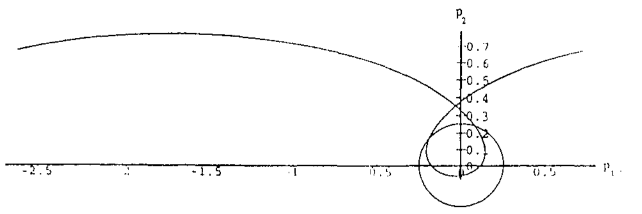

Fig. 1a Primer vector and circle of maximum thrust magnitude: $x_1(0) = 0, x_2(0) = 0.2k, x_1'(0) = x_2'(0) = 0, \theta_f = 2\pi, b = 0.235k$.

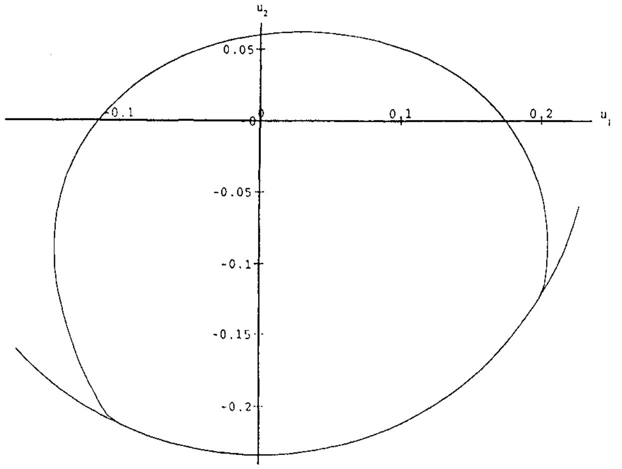

Fig. 1b Optimal thrust: $x_1(0) = 0, x_2(0) = 0.2k, x_1'(0) = x_2'(0) = 0, \theta_f = 2\pi, b = 0.235k$.

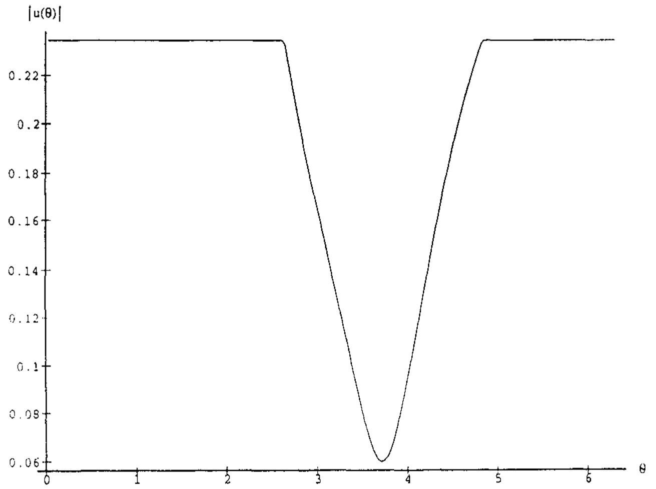

Fig. 1c Thrust magnitude vs θ: $x_1(0) = 0, x_2(0) = 0.2k, x_1'(0) = x_2'(0) = 0, \theta_f = 2\pi, b = 0.235k$.

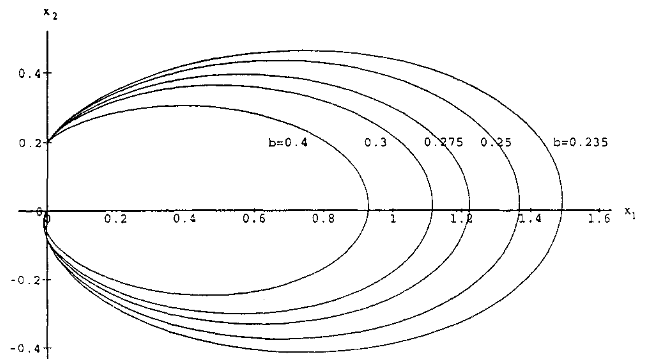

Fig. 2 Flight paths for various bounds on thrust magnitude: $x_1(0) = 0, x_2(0) = 0.2k, x_1'(0) = x_2'(0) = 0, \theta_f = 2\pi$.

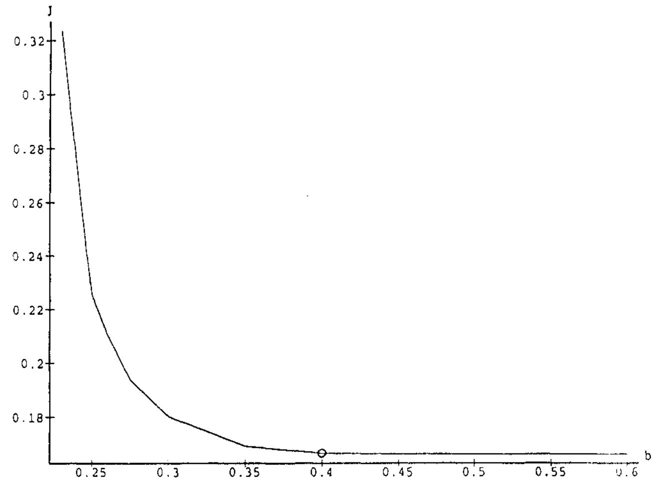

Fig. 3 Fuel cost vs bound on thrust magnitude: $x_1(0) = 0, x_2(0) = 0.2k, x_1'(0) = x_2'(0) = 0, \theta_f = 2\pi$.

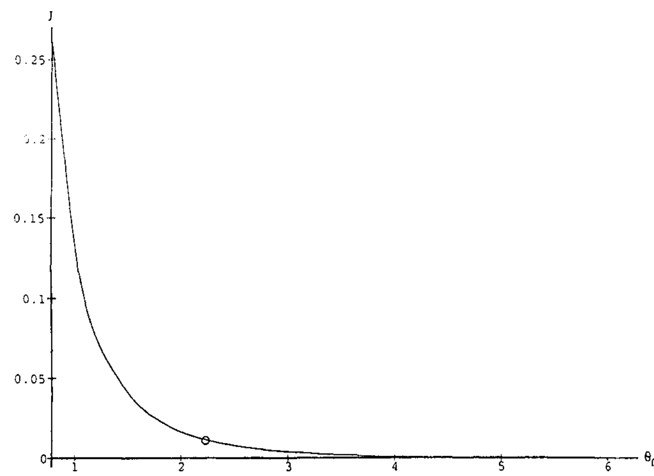

Fig. 4 Fuel cost vs flight duration: $x_1(0) = 0.1k, x_2(0) = 0, x_1'(0) = x_2'(0) = 0, b = 1.0k$.

rendezvous flight paths for various values of the saturation bound b. For values of b greater than 0.4 the solution is fully unsaturated, and the optimal rendezvous solution does not change. At $b = 0.235$ we have the outermost flight path in Fig. 2. This path is associated with the curves in Fig. 1 and is not far from the lower saturation limit that occurs near $b = 0.230$, where the solution is fully saturated, abnormal, and below which the boundary conditions cannot be satisfied on the interval $0 \leq \theta \leq 2\pi$. The interval $0.23 \leq b \leq 0.40$ therefore defines all possible flight paths for the given boundary conditions. The cost as a function of b is seen in Fig. 3. As b is lowered, the first evidence of saturation is seen around $b = 0.4$, the curve increasing more rapidly as b decreases below this level. For values of b greater than 0.4, the cost is constant since the solution is fully unsaturated in these cases.

Figure 4 demonstrates the results of a study of the effects of flight duration on cost for a fixed bound of the thrust magnitude. In this problem, the spacecraft begins 0.1 normalized units behind the satellite with zero initial velocity and a fixed bound of $b = 1.0$ normalized units and performs an optimal rendezvous with the satellite at the origin for various fixed values of θ_f. For values of θ_f less than approximately 2.3 rad, the cost increases sharply. This figure shows that significant savings in fuel can be obtained if the length of the flight interval can be increased.

Remark. The reader should note that y_f is zero for these simulations; consequently z_f is independent of the flight time, as is evident from Eq. (7). For other terminal conditions, z_f varies with flight time according to Eq. (7). Trajectory computations, in general, therefore, require recomputation of z_f whenever the flight interval is changed.

C. Application of Efficiency Condition

We demonstrate the application of the efficiency condition (27) for the rendezvous problem illustrated by Fig. 1.

The left side of Eq. (27) is the peak of the magnitude of the primer vector for an unsaturated solution of the problem. As is typical, this peak value occurs at the left end point. Its magnitude is calculated at approximately 0.453. Since $b = 0.235$ for this problem, the efficiency condition (27) is violated. Dividing the former number by the latter, we obtain approximately the number 1.92766, which is the right side of inequality (33). The smallest integer satisfying the inequality (33) is $\nu = 2$. We therefore require two thrusters to perform the rendezvous maneuver without saturation.

Using two thrusters, the effective bound becomes $b = 2(0.235) = 0.470$. Using this number, the graph in Fig. 3 indicates a cost of approximately $J = 0.166$ for the two thrusters. If this rendezvous is performed with only one thruster, the cost of this highly saturated rendezvous is $J = 0.323$, nearly twice as much.

If the two thrusters are used, the problem is solved in closed form, each of the two thrusters satisfying Eq. (34) with $\nu = 2$. The optimal psuedostate trajectory (8) then becomes

$$z(\theta) = M(\theta_0, \theta)M(\theta_0, \theta_f)^{-1}z_f$$

The original trajectory is then obtained by solving Eq. (6) for $y(\theta)$, getting

$$y(\theta) = \Psi(\theta)^{-T}\left[\Psi(\theta_0)y_0 + M(\theta_0, \theta)M(\theta_0, \theta_f)^{-1}z_f\right]$$

For this problem, $M(\theta_0, \theta)$ and $M(\theta_0, \theta_f)^{-1}$ have been evaluated in previous work.[20]

IV. Conclusions

We have considered the effects of thrust saturation on optimal rendezvous of a spacecraft. We have shown in great generality that thrust saturation can only deteriorate fuel efficiency and can never improve it. Moreover, a higher thrust saturation bound is never worse than a lower one. These very general arguments, however, do not reveal any quantitative information. For the specific case of optimal power-limited spacecraft rendezvous near a satellite in circular orbit, quantitative results are obtained and indicate that thrust saturation leads to a serious degradation in fuel performance. This situation can be improved by lengthening the flight interval or implementing additional thrusters, if possible. For general linear equations of motion, we presented the existence condition and the efficiency condition. The computation involved in these conditions is not difficult and yields respective information on the attainability of a boundary point in a given time and whether or not saturation occurs with its consequent loss in performance. The efficiency condition can also be used to determine the minimum number of thrusters required to avoid saturation losses and to provide a closed-form solution for certain linear systems. This was illustrated by a computational example in which fuel expenditure was reduced by nearly 50% through the use of two thrusters instead of one.

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